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ACTIVE CONTROL OF NON-LINEAR PANEL VIBRATION AND SOUND RADIATION

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1. INTRODUCTION

In this note, the results are presented of some analytical and numerical studies on the active control of non-linear panel vibrations and sound radiation due to wall pressure fluctuation. In our previous work [1], we treated the problem of passive control of non-linear panel vibration by boundary damping without the wall pressure excitation. It was shown there that a slight boundary damping can result in an exponentially fast decay in vibrational energy. However, with a persistent excitation, the passive control is ineffective. It seems necessary to apply an active control force as an effective counter-measure. The main purpose of this note is to study, based on a non-linear panel model, the effectiveness of such control to suppress the panel vibration and sound radiation induced by the unsteady pressure forcing.

Without control, non-linear motion of elastic panels has been studied by E. H. Dowell [2, 3], Nayfeh and Mook [4] and others. Experimental and numerical studies of such non-linear interaction problems with and without control were carried out by Maestrello and his collaborators [5–7]. Analytically, optimal control of beams or panels with concentrated forces as actuators was treated by Su and Tadjbakhsh [8], by neglecting the non-linear tension term and the wall pressure excitation. However, so far, little analytical work on optimal control of non-linear panel vibrations and sound radiation has been undertaken.

In this note we deal with the forced vibration due to wall pressure alone. The control consists of a distributed force applied normally to one side of the wall. For simplicity, the flexible panel is assumed to be hinged to rigid plates at both ends (see the schematic diagram in Figure 1). In section 2, the coupled equations governing the non-linear panel vibration and acoustic radiation problem are given. The optimal control problem is formulated in section 3. For the optimality criterion, a time-average cost or objective functional is introduced to measure the performance in controlling the vibration and sound radiation. In section 4, by applying the variational method, we derive the optimality equation for the control force distribution which is coupled with the controlled equations of motion. By using an eigenfunction expansion, the modal control problem is discussed in section 5. Then we solve a truncated modal control problem numerically by the shooting method for a two-point boundary value problem in time domain. The numerical results are described in section 6 and shown in Figures 2–13. In the last section, the main results of the note are summarized and discussed to reach the conclusions of this study.

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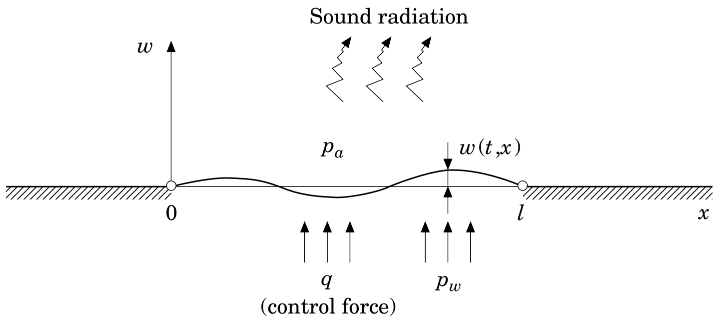

Figure 1. A schematic diagram for the control of sound and vibration.

2. GOVERNING EQUATIONS

We consider the non-linear vibration of a rectangular panel, the mid-plane of which is given by $0 \leqslant x \leqslant l$, $0 \leqslant y \leqslant d$. If $l \gg d$, the vertical deflection w is nearly uniform in the transverse direction y, and an one-dimensional structural model is often used. The most well-known equation for studying non-linear panel vibration or fluttering is given by the following partial (integro-)differential equation [2]:

$$m\frac{\partial^2 w}{\partial t^2} + \gamma\frac{\partial w}{\partial t} + [P - N(t)]\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^4 w}{\partial x^4} = \Delta p(t, x), \qquad 0 < x < l, \tag{1}$$

where m is the mass per unit length, γ is the damping coefficient, P is the compressive force, $D = Eh^3/12(1 - \nu^2)$ is the bending stiffness, and

$$N(t) = (Eh/2l)\int_0^l \left[\frac{\partial}{\partial x} w(t, x)\right]^2 \mathrm{d}x \tag{2}$$

is the tension induced by the mid-plane stretching. The constants E, h and ν denote Young's modulus, the panel thickness and the Poisson ratio, respectively. On the right side of equation (1), Δp denotes the pressure difference across the panel:

$$\Delta p = p_w - p_a, \tag{3}$$

where p_a and p_w are the wall pressures above and below the panel. As shown in the schematic diagram, p_w may be the wall pressure in a boundary layer flow or other acoustic loading. Also, we have

$$p_a = p_0 + p_1, \tag{4}$$

where p_0 is the (constant) ambient air pressure and p_1 is the acoustic pressure on the upper wall. The initial conditions for the panel are

$$w(0, x) = g(x), \qquad w_t(0, x) = h(x), \tag{5, 6}$$

where w_t and w_x denote the partial derivatives in t and x, and so forth. For a simply supported panel, the boundary conditions are

$$w(t, 0) = w(t, l) = 0, \qquad w_{xx}(t, 0) = w_{xx}(t, l) = 0. \tag{7, 8}$$

Other boundary conditions, such as the clamped ends or the periodic boundary conditions, may also be prescribed.

In the upper plane $y > 0$, the fluid is assumed to be an inviscid ideal gas and the flow is irrotational. Then the acoustic velocity components u and v in the x and y directions, and the acoustic pressure p, are derivable from a potential $\phi(t, x, y)$ such that

$$u = \phi_x, \qquad v = \phi_y, \tag{9}$$

and

$$p = (1/c^2)\phi_t(t, x, y), \tag{10}$$

where c is the ambient speed of sound. Moreover, the amplitude of sound emission is governed by the wave equation,

$$\phi_{tt} = c^2(\phi_{xx} + \phi_{yy}), \qquad y > 0, \tag{11}$$

for $-\infty < x < \infty$ and $y > 0$. Here the linear acoustic equation is justifiable, since the amplitude of panel vibration is small compared with the acoustic wavelength. Initially, the air is assumed to be quiescent, so that

$$\phi(0, x, y) = \phi_t(0, x, y) = 0. \tag{12}$$

On the flexible boundary, we have the panel-flow coupling condition:

$$\phi_y(t, x, 0) = w_t(t, x), \qquad 0 < x < l. \tag{13}$$

On the rigid wall, the normal velocity $v = 0$, so that

$$\phi_y(t, x, 0) = 0 \qquad \text{for } x \leqslant 0 \quad \text{or} \quad x \geqslant l. \tag{14}$$

Note that, in equation (4), the acoustic wall pressure p_1 is related to p by

$$p_1(t, x) = p(t, x, 0). \tag{15}$$

Given the wall pressures p_w and p_0, together with the suitable initial boundary conditions, equations (1) and (11) must be solved to determine the panel vibration and the acoustic radiation field.

3. FORMULATION OF CONTROL PROBLEM

We assume that the wall pressures p_w and p_a can be measured so that the pressure differential $\Delta p(t, x)$ in equation (1) is known. To suppress the panel vibration and sound generation, a control force or pressure $q(t, x)$ is applied to regulate the panel motion. Then the controlled equation reads

$$mw_{tt} + \gamma w_t + [P - N(t)]w_{xx} + Dw_{xxxx} = \Delta p(t, x) + q(t, x), \qquad 0 < x < l, \tag{16}$$

which is subject to the same initial and boundary conditions (5)–(8).

To reduce noise and vibration, the control objective is to minimize the performance index measuring the overall level of vibration or radiation intensity with a limited control effort. For the control of panel vibration, we propose the objective function

$$J(q) = (1/2T)\int_0^T \int_0^l \{w_t^2 + bw_{xx}^2 + cw^2 + kq^2\}\, \mathrm{d}t\, \mathrm{d}x, \tag{17}$$

where b, c and k are given positive constants. J represents the time average of a weighted combination of the kinetic energy, the flexural potential energy, the vibration amplitude and the control cost. The weighting factors b, c and k are assigned subjectively according to the priority in attaining the control objective.

On physical grounds, it seems clear that the controlled panel vibration will also result in a reduced sound radiation. For the sound radiation control, a more pertinent criterion is to replace the vibrational energy in the objective function (17) by the mean acoustic radiation energy imparted by the panel:

$$J_1(q) = (1/T)\int_0^T\int_0^l \{pv|_{y=0} + \tfrac{1}{2}kq^2\}\,\mathrm{d}t\,\mathrm{d}x. \tag{18}$$

However, in this note, the optimal control criterion will be based solely on the objective function (17).

4. DERIVATION OF OPTIMAL CONTROL EQUATIONS

In this section, we shall derive the optimality equations by the method of variational calculus [9]. We seek to minimize the objective function $J(q)$ in equation (17) for the panel vibration problem with respect to the control of q under the constraint by the dynamical (state) equation (16). For J being minimal, it is necessary for its first variation δJ of equation (17):

$$\delta J(q) = (1/T)\int_0^T\int_0^l \{w_t\delta w_t + bw_{xx}\delta w_{xx} + cw\delta w + kq\delta q\}\,\mathrm{d}t\,\mathrm{d}x = 0. \tag{19}$$

To rid the variation δw of its derivatives in the above integral, we integrate by parts as follows:

$$\int_0^T\int_0^l w_t\delta w_t\,\mathrm{d}t\,\mathrm{d}x = \int_0^l w_t(T,x)\delta w(T,x)\,\mathrm{d}x - \int_0^T\int_0^l w_{tt}\delta w\,\mathrm{d}t\,\mathrm{d}x \tag{20}$$

and

$$\int_0^T\int_0^l w_{xx}\delta w_{xx}\,\mathrm{d}t\,\mathrm{d}x = \int_0^T\int_0^l w_{xxxx}\delta w\,\mathrm{d}t\,\mathrm{d}x, \tag{21}$$

where use was made of the boundary conditions for w and the fact that $\delta w = 0$ at $t = 0$, $x = 0, l$. In equation (19), to eliminate δq in favor of δw, we take the variation of the equation (16), to obtain

$$\delta q = \mathscr{L}\delta w - w_{xx}\delta N(t), \tag{22}$$

where

$$\mathscr{L}\varphi = m\varphi_{tt} + \gamma\varphi_t + [P - N(t)]\varphi_{xx} + D\varphi_{xxxx}. \tag{23}$$

Now, in view of equation (2), we obtain

$$\delta N(t) = \eta\int_0^l w_x\delta w_x\,\mathrm{d}x,$$

so that equation (22) can be written as

$$\delta q = \mathscr{L}\delta w - \eta w_{xx}\int_0^l w_x\delta w_x\,\mathrm{d}x, \qquad \text{with } \eta = (Eh/l). \tag{24}$$

It follows from equation (24) that

$$\int_0^T \int_0^l q\delta q \, \mathrm{d}t \, \mathrm{d}x = \int_0^T \int_0^l q\{\mathscr{L}\delta w - \eta w_{xx} \int_0^l w_r \delta w_r \, \mathrm{d}r\} \, \mathrm{d}t \, \mathrm{d}x. \tag{25}$$

Again by integrating by parts several times, we obtain

$$\int_0^T \int_0^l q(\mathscr{L}\delta w) \, \mathrm{d}t \, \mathrm{d}x = \int_0^T \int_0^l (\mathscr{L}q)\delta w \, \mathrm{d}t \, \mathrm{d}x - 2\gamma \int_0^T \int_0^l q_t \delta w \, \mathrm{d}t \, \mathrm{d}x - m \int_0^l (q_t \delta w)|_{t=T} \, \mathrm{d}x \tag{26}$$

and

$$\int_0^T \int_0^l q w_{xx} \left(\int_0^l w_r \delta w_r \, \mathrm{d}r \right) \mathrm{d}t \, \mathrm{d}x = - \int_0^T \int_0^l w_{xx} \left(\int_0^l q w_{rr} \, \mathrm{d}r \right) \delta w \, \mathrm{d}t \, \mathrm{d}x, \tag{27}$$

provided that the following condtions hold:

$$\delta w = \delta w_t = 0 \quad \text{at } t = 0, \qquad \delta w = \delta w_{xx} = 0 \quad \text{at } x = 0, l$$

and

$$q = 0 \quad \text{at } t = T \qquad \text{and} \qquad q = q_{xx} = 0 \quad \text{at } x = 0, l. \tag{28}$$

By substituting equations (26) and (27) into equation (25), yields

$$\int_0^T \int_0^l q\delta \, \mathrm{d}t \, \mathrm{d}x = \left\{ \mathscr{L}q - 2\gamma q_t + \eta w_{xx} \left(\int_0^l q w_{rr} \, \mathrm{d}r \right) \right\} \delta w \, \mathrm{d}t \, \mathrm{d}x - m \int_0^l (q_t \delta w)|_{t=T} \, \mathrm{d}x. \tag{29}$$

By virtue of equations (20), (21) and (29) and setting $\delta J = 0$, equation (19) gives rise to the variational equation:

$$\int_0^T \int_0^l k \left\{ \mathscr{L}q - 2\gamma q_t + \eta w_{xx} \left(\int_0^l q w_{rr} \, \mathrm{d}r \right) - (1/k) mw \right\} \delta w \, \mathrm{d}t \, \mathrm{d}x$$

$$= m \int_0^l \{k q_t - w_t\} \delta w|_{t=T} \, \mathrm{d}x, \tag{30}$$

where

$$\mathscr{M}w = (w_{tt} - k w_{xxxx} - cw). \tag{31}$$

From equations (30) and (28), we deduce the equation for the optimal control:

$$\mathscr{L}q - 2\gamma q_t = (1/k)\mathscr{M}w, \tag{32}$$

or

$$m q_{tt} - \gamma q_t + [P - N(t)] q_{xx} + D q_{xxxx} + \eta \left(\int_0^l w_{rr} q \, \mathrm{d}r \right) w_{xx} = (1/k)(w_{tt} - b w_{xxxx} - cw), \tag{33}$$

which is subject to the terminal and boundary conditions

$$q(T, x) = 0, \qquad q_t(T, x) = (1/k)w_t(T, x) \tag{34}$$

and

$$q(t, 0) = q(t, l) = 0, \qquad q_{xx}(t, 0) = q_{xx}(t, l) = 0. \tag{35}$$

The coupled equations (16) and (33) together with the appropriate side conditions form the optimality system, and their solutions give the optimal state w^* and the optimal control q^*, respectively.

It is well known that the solution of (11) and (12) can be expressed as

$$\phi(t, x, y) = \int_0^t \int_0^l G(t-s, x-\xi, y) w_s(s, \xi)\, \mathrm{d}s\, \mathrm{d}\xi, \tag{36}$$

where G is a Green (or Riemann) function for the half-space problem, which can be obtained easily from that of the free space by the method of images [10]:

$$G(t, x, y) = (1/\pi)\{t^2 - (x^2 + y^2)\}^{-1/2} \mathrm{H}\{t^2 - (x^2 + y^2)\}, \tag{37}$$

where the Heaviside function $\mathrm{H}(\tau) = 1$ for $\tau \geqslant 0$ and $=0$ for $\tau < 0$. In view of equations (9), (10) and (13), equation (18) can be rewritten as

$$J_1(q) = (-1/c^2T) \int_0^T \int_0^t \int_0^l \int_0^l G_t(t-s, x-\xi, 0) w_t(t, x) w_s(s, y)\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}\xi$$

$$+ (k/2T) \int_0^T \int_0^l q^2(t, x)\, \mathrm{d}t\, \mathrm{d}x, \tag{38}$$

where $G_t = \partial/\partial t G$ is a generalized derivative. Similar to the derivation of the optimal control equation (33) for vibration, the optimality equation for the minimum q^* of the objective function (38) can be obtained, but for brevity will not be written down.

5. MODAL CONTROL PROBLEM

For the linear vibration problem, the associated eigenfunctions are given by

$$\varphi_j(x) = \sin(j\pi x/l), \qquad j = 1, 2, \ldots, n, \ldots. \tag{39}$$

As the n-mode approximation to the control of panel vibration, we expand the solution w, the wall pressure Δp and the inital functions g and h into sine series truncated at the nth term, to give

$$w(t, x) \sim \sum_{j=1}^n w_j(t)\varphi_j(x), \qquad \Delta p(t, x) \sim \sum_{j=1}^n p_j(t)\varphi_j(x), \tag{40, 41}$$

$$g(x) \sim \sum_{j=1}^n g_j\varphi_j(x), \qquad h(x) \sim \sum_{j=1}^n h_j\varphi_j(x), \tag{42, 43}$$

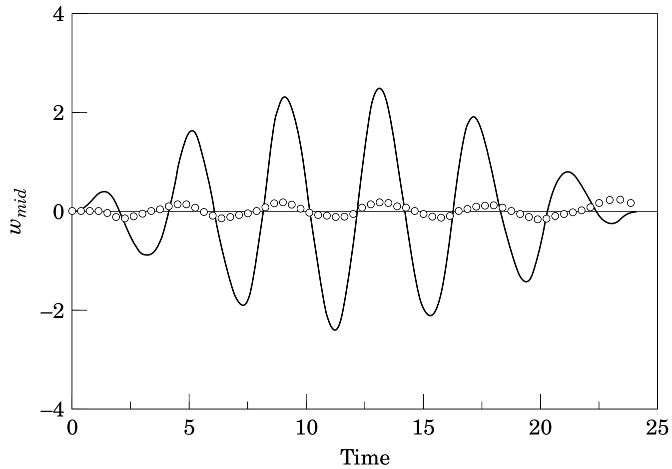

Figure 2. The mid-point deflection versus time with and without control (linear case): ——, uncontrolled; ○○○○○, controlled.

and the corresponding modal control assumes the form

$$q(t, x) \sim \sum_{j=1}^{n} q_j(t)\varphi_j(x). \tag{44}$$

By substituting equations (40)–(44) into equation (16) and the conditions (5) and (6), we obtain

$$m\ddot{w}_j + \gamma\dot{w}_j + \lambda_j^2 w_j + Q_{nj}(w)w_j = p_j(t) + q_j(t), \tag{45}$$

$$w_j(0) = g_j, \quad \dot{w}_j(0) = h_j, \qquad j = 1, 2, \ldots, n, \tag{46}$$

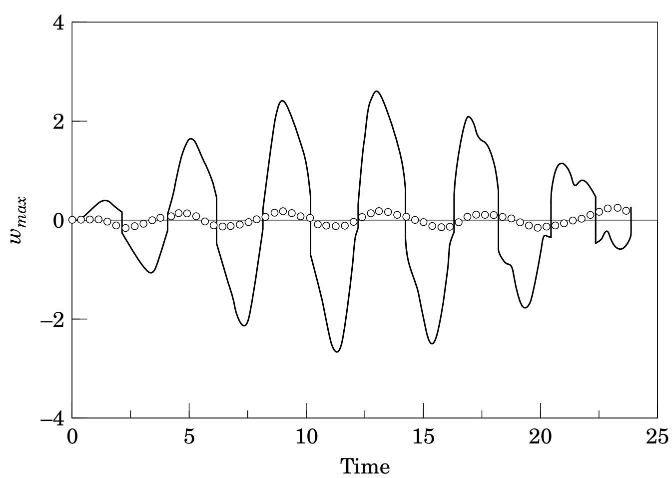

Figure 3. The maximum deflection versus time with and without control (linear case): ——, uncontrolled; ○○○○○, controlled.

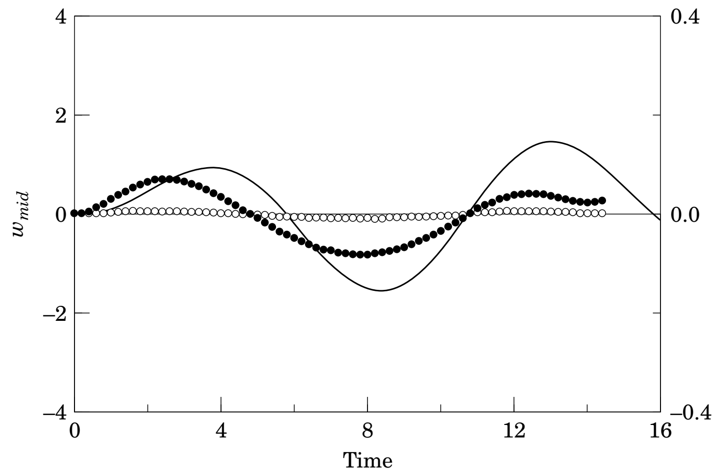

Figure 4. The mid-point deflection versus time with and without control (non-linear case): ——, uncontrolled; ○○○○○, controlled; ●●●●●, controlled and magnified ten times.

where

$$\lambda_j^2 = [D(j\pi/l)^4 - P(j\pi/l)^2], \qquad Q_{nj}(w) = \eta(j\pi/l)^2 \sum_{i=1}^{n} (i\pi/l)^2 w_i^2. \tag{47, 48}$$

In view of equations (39) and (40), equation (33) and the terminal conditions (34) yield

$$m\ddot{q}_j - \gamma\dot{q}_j + \lambda_j^2 q_j + Q_{nj}(w)q_j = (1/k)(\ddot{w}_j - A_j w_j), \tag{49}$$

$$q_j(T) = 0, \quad \dot{q}_j(T) = (1/k)\dot{w}_j(T), \qquad j = 1, 2, \ldots, n, \tag{50}$$

where

$$A_j = b(j\pi/l)^4 + c. \tag{51}$$

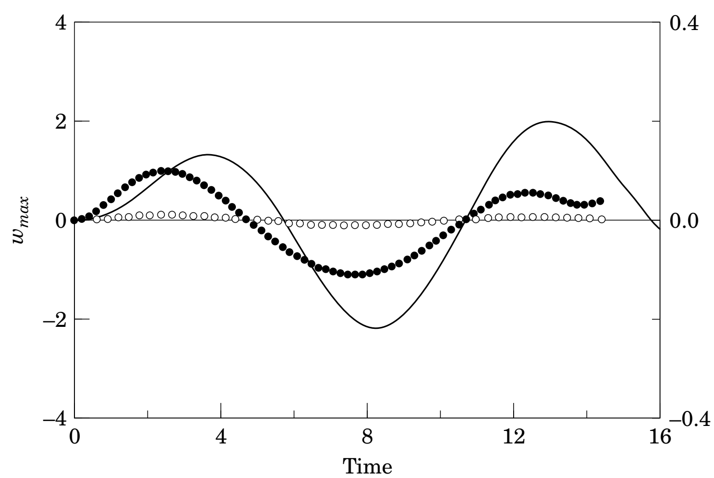

Figure 5. The maximum deflection versus time with and without control (non-linear case): ——, uncontrolled; ○○○○○, controlled; ●●●●●, controlled and magnified ten times.

For the above modal control problem, the objective function J in equation (17) can be evaluated as

$$J(q) = (l/2T)\int_0^T \sum_{j=1}^{n} \{\dot{w}_j^2 + [b(j\pi/l)^2 + c]w_j^2 + kq_j^2\}\,\mathrm{d}t. \qquad (52)$$

The corresponding J_1 in equation (38) for the radiation problem gives

$$J_1(q) = (1/c^2T)\int_0^T\int_0^t \sum_{i,j=1}^{n} g_{ij}(t-s)\dot{w}_i(t)\dot{w}_j(s)\,\mathrm{d}s\,\mathrm{d}t + (l/2T)\int_0^T \sum_{j=1}^{n} q_j^2(t)\,\mathrm{d}t, \qquad (53)$$

where

$$g_{ij}(t) = -\int_0^l\int_0^l G_t(t, x-\xi, 0)\varphi_i(x)\varphi_j(\xi)\,\mathrm{d}x\,\mathrm{d}\xi. \qquad (54)$$

In what follows, the optimality system (45) and (49) for the vibration control problem will be solved numerically. Using these results as a sub-optimal radiation control, the sound radiation power will be computed.

6. NUMERICAL RESULTS

To solve the truncated modal control numerically, we note that the coupled, non-linear optimality system (45) and (49) is a two-point boundary value problem, instead of an initial valued problem. In addition to the large size of the system, the non-linear boundary value

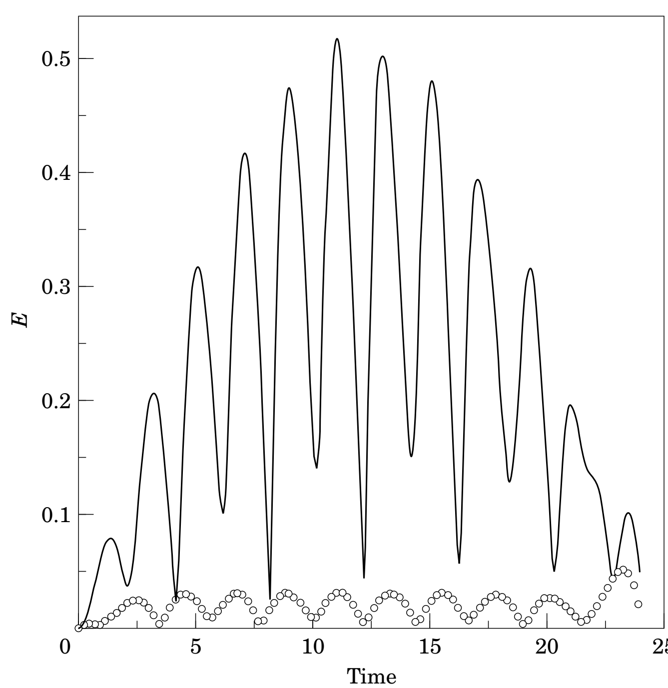

Figure 6. The Euclidean norm of the mid-point amplitudes of all truncated modes (linear case): ——, uncontrolled; ○○○○○, controlled.

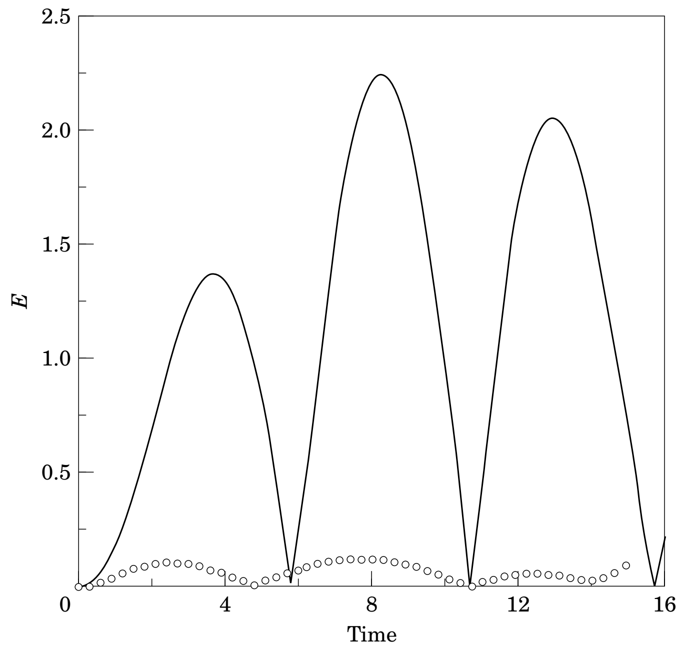

Figure 7. The Euclidean norm of the mid-point amplitudes of all truncated modes (non-linear case): ——, uncontrolled; ○○○○○, controlled.

problem is difficult to solve numerically because of lack of stability. As a result, we can handle only a small set of modal equations. For computational accuracy and efficiency, we adopted the shooting method [11], combined with the modified Newton's method and the fourth order Runge–Kutta scheme [12]. The results are shown in Figures 2–13. Except for Figures 10–12, the results were computed by choosing the following set of parameter

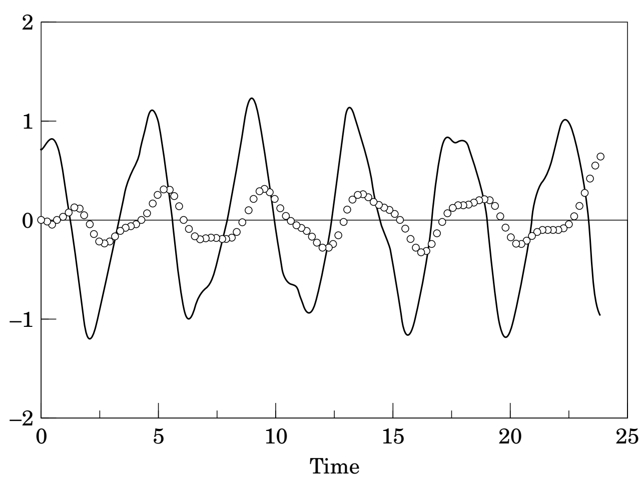

Figure 8. The amplitudes of control force and external force histories (linear case): ——, external force; ○○○○○, control force.

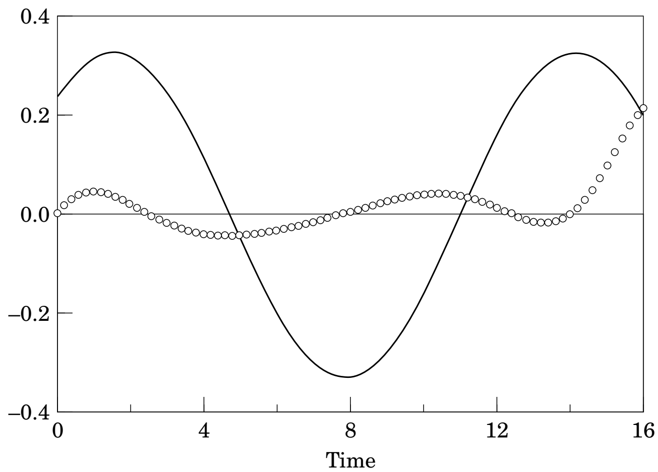

Figure 9. The amplitudes of control force and external force histories (non-linear case): ——, external force; ○○○○○, control force.

values: $l = 4$, $g_j = h_j = 0$, $m = 1$, $b = 0{\cdot}1$, $c = 1$, $\gamma = 0{\cdot}01$, $D = 0{\cdot}02$, $k = 1$, the time interval $T = 16$–25, and

$$p_j(t) = (1/3j)(\cos t/2 + \sin t/2), \qquad j = 1,2, \ldots, n.$$

For Figures 10–12, we increased the stiffness D from 0·02 to 1, 10 and 100, respectively. We first neglected the non-linear term by setting $Q_{nj} = 0$ in equations (45) and (49). The results for the mid-point and the maximum panel deflections, and the Euclidean norm E of the mid-point amplitudes of all modes with and without controls are plotted in Figures 2, 3 and 6, respectively. They show that the active control is very effective in suppressing vibrations for the linear case, where the norm E is given by $\{\Sigma_{j=1}^{n} w_j^2(t)\varphi_j^2(2)\}^{1/2}$. The corresponding results for the non-linear case are shown in Figures 4, 5 and 7. There we see that the control is equally effective. In Figures 4 and 5, for visualization, the controlled deflections are magnified ten times to the scale along the right ordinate.

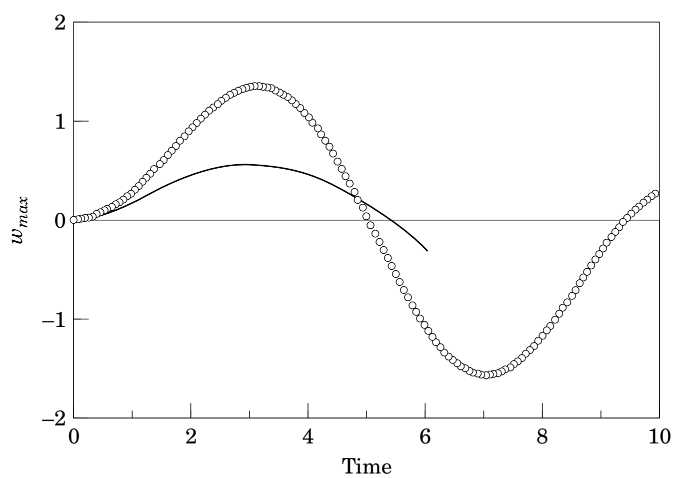

Figure 10. The maximum deflection versus time with and without control at high frequencies (non-linear case, $D = 1$): ——, controlled; ○○○○○, uncontrolled.

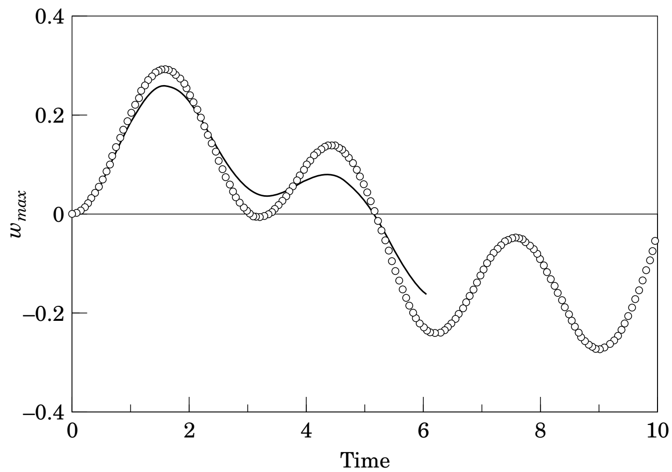

Figure 11. The maximum deflection versus time with and without control at high frequencies (non-linear case, $D = 10$): ——, controlled; ○○○○○, uncontrolled.

However, as the stiffness or the vibration frequencies increases, the system respond less sensitively to the control, much more so for the non-linear case. As shown in Figures 10–12, changing $D = 0{\cdot}02$ to 1, 10 and 100, the non-linear system quickly runs out of control. To see how the control works, the histories of the external and the control forces for the linear and non-linear cases are displayed in Figures 8 and 9, respectively. Finally, the optimally controlled vibration results were used to compute the sound radiation intensity $I = J_1$ with $k = 0$. The ratio I/I_0 of the controlled and the uncontrolled sound radiation intensities versus the forcing amplitude is plotted in Figure 13. This shows that, by using the optimal control for the vibration problem, the sound intensity can also be reduced significantly, even more so for the linear case. The numerical results seem to support our prediction that the vibrational control may also be used as an effective, sub-optimal sound radiation control.

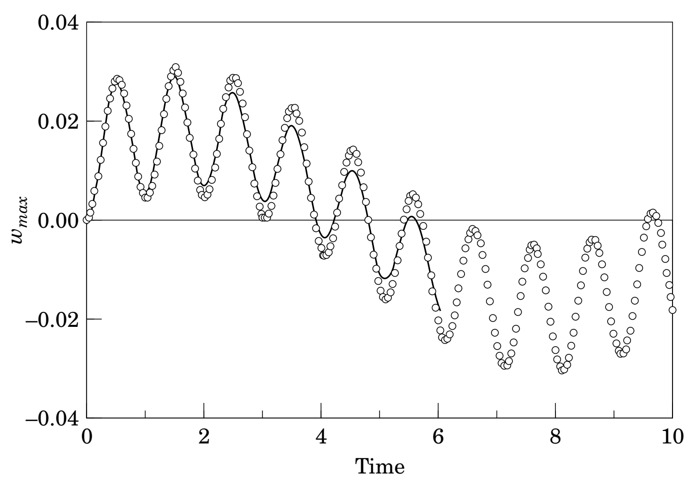

Figure 12. The maximum deflection versus time with and without control at high frequencies (non-linear case, $D = 100$): ——, controlled; ○○○○○, uncontrolled.

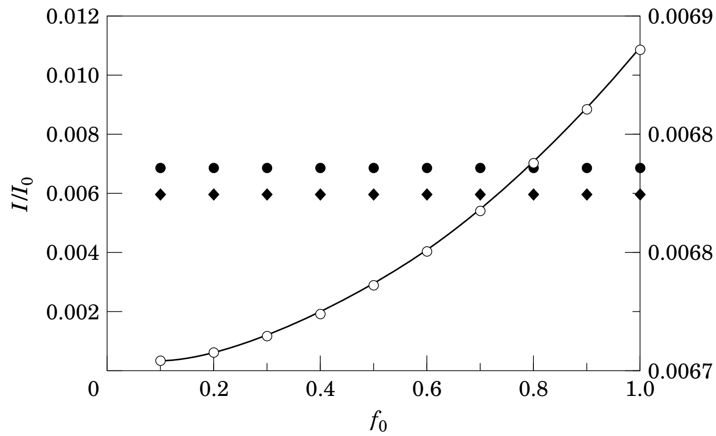

Figure 13. The sound radiation intensity ratio versus the maximum amplitude of the external force f_0 (I_0 = uncontrolled intensity): ◆◆◆◆◆, linear; ●●●●●, non-linear; ○○○○○, non-linear and magnified.

7. CONCLUSIONS

By the variational method, the active control of sound and vibration of a non-linear elastic panel excited by wall pressure fluctuation was studied analytically and numerically. The main results of this note are summarized and discussed as follows.

(1) For the control of panel vibration, given the control objective function (17), the optimal control can be found in the form of an external pressure applied to the wall. We derived the partial differential equation (33) for the optimal control which is coupled to the equation of motion (16).

(2) For the given objective function (18), although omitted for brevity, it is possible to derive the optimality system for the control of sound radiation governed by the wave equation.

(3) In both cases, the optimality system consists of a coupled non-linear boundary value problem in space and time.

(4) In the case of truncated modal control, the optimality system yields a two-point boundary value problem for a finite set of non-linear ordinary differential equations (45) and (49).

(5) The truncated optimality system for modal control was solved numerically. The results show the following: (i) for linear panel vibration, the control is highly effective and can almost completely eliminate the vibration over a short time horizon. (ii) In contrast, a non-linear panel, in general, responses less sensitively to the active control. The control is more effective at lower vibration frequencies and with weak non-linearity. For fixed non-linearity, the effectiveness of control diminishes as the frequencies increase, and eventually the system loses control completely. (iii) By applying the optimal vibration control, the sound radiation intensity can also be reduced significantly.

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